Problem Set 3 due September 25, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1:

(1) Given numbers a and b, for which number c does the system:

$$\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

have a solution v_1, v_2. *(5 points)*

(2) Draw the set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfing the conditions in part (1) on a picture of $\mathbb{R}^3$. *(5 points)*

(3) Construct a 3×4 matrix whose column space is generated by $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$. *(5 points)*

Problem 2: Let us consider an $m \times n$ matrix R with block decomposition:

$$R = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} \tag{1}$$

where I is a (square) unit matrix. So in other words, R is in reduced row echelon form with all the pivot columns to the left of the free columns. Let r be the rank of R.

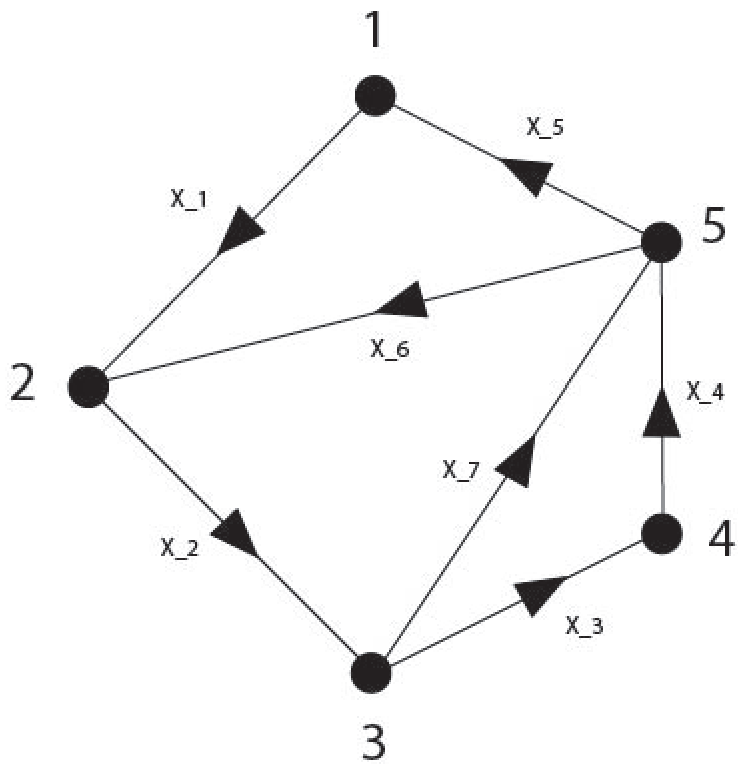
(1) What are the number of rows and columns of each of the 4 blocks in (1), in terms of m, n and r? *(10 points)*

(2) If $r = m$, find a right-inverse for R, i.e. a matrix Q such that $RQ = I_m$ *(5 points)*

(3) If $r = n$, find a left-inverse for R, i.e. a matrix Q such that $QR = I_n$ *(5 points)*

(4) If you are allowed to perform column operations on R (i.e. adding arbitrary multiples of any column to any other column), then what is the simplest form to which you can bring R? *(5 points)*

Problem 3: The diagram below represents 5 nodes (represented by the circles) connected by 7 pieces of conducting wire (represented by the lines). The intensity of the current flowing through these pieces of wire is $x_1, ..., x_7$, in the direction of the arrow. If any of the x_i's are negative, this just means current flowing in opposite direction to the arrow.



Kirchoff's first law says that, at every node, the incoming current should equal the outgoing current.

(1) Write down explicitly the incidence matrix of the diagram. By definition, this is the 5×7 matrix A whose entry at row i and column j is:

$$\begin{cases} 1 & \text{if the current on the } j\text{–th wire flows into node } i \\ -1 & \text{if the current on the } j\text{–th wire flows out of node } i \\ 0 & \text{if the } j\text{–th wire does not intersect node } i \end{cases}$$

(the j–th wire is the one denoted by the variable x_j in the diagram). *(5 points)*

(2) Express Kirchoff's first law as a linear algebra condition on the vector of currents $\begin{bmatrix} x_1 \\ ... \\ x_7 \end{bmatrix}$, which involves the incidence matrix A (justify). *(5 points)*

(3) By using the row reduced echelon form of A, find all possible vectors of currents $\begin{bmatrix} x_1 \\ ... \\ x_7 \end{bmatrix}$ which satisfy Kirchoff's law for the diagram above. *(10 points)*

Problem 4: (justify all your answers)

(1) If X is an invertible square matrix, what can you say about $C(X)$ and $N(X)$? *(10 points)*

(2) If $Y = \begin{bmatrix} A \\ B \end{bmatrix}$ is a block matrix, what is $N(Y)$ in terms of $N(A)$ and $N(B)$? *(5 points)*

(3) If $Z = \begin{bmatrix} A & B \end{bmatrix}$ is a block matrix, what is $C(Z)$ in terms of $C(A)$ and $C(B)$? *(5 points)*

Problem 5:

(1) Compute the reduced row echelon form of the matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & -3 \\ 2 & 4 & 0 & -6 \end{bmatrix}$$

(all zero rows should be at the bottom of A). *(10 points)*

(2) Use the result of part (1) to find the full set of solutions to the equation:

$$A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

(10 points)